J. Fluid Mech. (1981), *vol.* 109, *pp.* 115–123
Printed in Great Britain

Diffraction of gravity waves by a barrier reef

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(Received 16 July 1980 and in revised form 6 November 1980)

The gravity-wave scattering matrix for a barrier reef that separates two different depths of water is calculated by an extension of a variational analysis of diffraction by a discontinuity in depth (Miles 1967). The resulting transmission coefficient for equal depths differs from that obtained by Johnson, Fuchs & Morison (1951), which appears to be incorrect. The results are applied to the calculation of resonant amplification of incoming swell or tsunamis by a shallow lagoon that is bounded by the reef and a vertical inner boundary. The results for this last problem agree closely (exactly for equal depths) with those obtained by Tuck (1980) through a rather different approach.

1. Introduction

I consider here the diffraction of gravity waves by a thin barrier that separates two domains of uniform depths h_1 $(x < 0)$ and h_2 $(x > 0)$, $h_2 > h_1$; see figure 1. The termination of the shallower domain by a vertical cliff at $x = -d$ yields a model for the partial trapping and amplification of waves between a coastline and a parallel reef (Allison & Grassia 1979; Tuck *et al.* 1980). Perhaps the most important natural problem is that of tsunami amplification, for which $Kh_1 < Kh_2 \ll 1$, where $K \equiv \sigma^2/g$ and σ is the angular frequency. Surf amplification in a shallow lagoon ($Kh_1 \ll 1$) with deep water ($Kh_2 \gg 1$) outside of the reef also is of practical interest.

The basic analysis follows that for a discontinuity in depth with no projecting barrier (Miles 1967; hereinafter referred to as I, followed by the appropriate equation or section number). The notation follows that in I, and much of the analytical detail, which is essentially similar to that in I, is omitted. The present results reduce to those of I in the limit $a/h_1 \uparrow 1$. The domain $-d < x < 0$ may act as a Helmholtz resonator if $a/h_1 \ll 1$ (cf. Miles & Munk 1961), and the limit $a/h_1 \downarrow 0$ is singular (the domains $x < 0$ and $x > 0$ are independent if the top of the barrier is not submerged).

I begin, in §2, by stating the boundary-value problem and introducing the scattering matrix that relates the amplitudes of the incoming and outgoing waves when the vertical boundary at $x = -d$ is absent. I then, in §3, obtain variational approximations to the elements of the scattering matrix. I consider the special case of equal depths in §4 and find that the transmission coefficient for the barrier differs from that obtained by Johnson *et al.* (1951). Finally, in §5, I impose the boundary condition at $x = -d$ ($d \gg h_1$) and obtain the resonant frequency of the dominant mode and the corresponding Q.

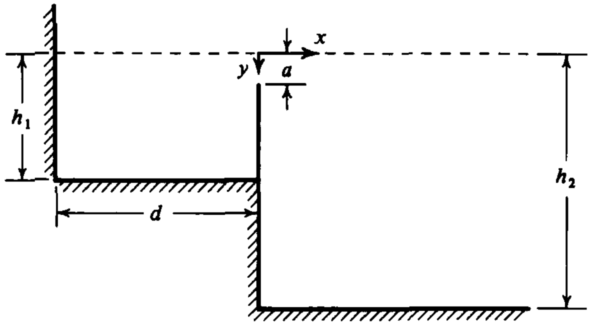

FIGURE 1. Sketch of barrier-reef model with vertical cliff at $x = -d$.

2. The scattering problem

The particle velocity and free-surface displacement, on the assumption of irrotational, incompressible flow, are given by

$$\mathbf{v} = \mathrm{Re}\,[e^{-i\sigma t}\nabla\phi(x,y)], \quad \eta = \mathrm{Re}\,[i(\sigma/g)\,e^{-i\sigma t}\phi(x,0)], \tag{2.1a,b}$$

where σ is the angular frequency and ϕ is the velocity potential, which satisfies

$$\nabla^2\phi = 0. \tag{2.2}$$

The linearized boundary conditions are (see figure 1)

$$\phi_y + K\phi = 0 \quad (y = 0), \quad \phi_y = 0 \quad (y = h), \tag{2.3a,b}$$

$$\phi_x = 0 \quad (x = 0,\ +a < y < h), \quad \phi_x = 0 \quad (x = -d,\ 0 < y < h), \tag{2.4a,b}$$

where $K = \sigma^2/g$, and the subscripts x and y signify partial differentiation; (2.3b) must be replaced by an appropriate finiteness condition for $x > 0$ in the limit of infinite depth ($h_2 \to \infty$). The asymptotic form of the solution at large distances from the discontinuity at $x = 0$ (at which non-propagated modes are excited) may be posed in the form [see I(2.10)]

$$\phi \sim (Ae^{-i\kappa|x|} + Be^{i\kappa|x|})\,\chi(y)\,\mathrm{sgn}\,x \quad (|x| \gg h), \tag{2.5}$$

where
$$\chi(y) = 2^{\frac{1}{2}}(h + K^{-1}\sinh^2\kappa h)^{-\frac{1}{2}}\cosh[\kappa(h-y)] \tag{2.6}$$

and
$$\kappa\tanh\kappa h = K. \tag{2.7}$$

The subscript $m = 1$ (2) is appended to h, κ, A, B, ϕ and χ in the sequel to signify $x < (>)\,0$.

The solution of the scattering problem posed by (2.2), (2.3a, b), (2.4a) and (2.5) may be expressed in terms of the aperture velocity ϕ_x ($x = 0$, $0 < y < a$) and parallels that of I, with the upper limit of integration in I(2.12), I(2.13), I(2.17), I(3.4), I(3.5), I(3.9), I(3.10) replaced by a (the present problem reduces to that of I for $a = h_1$). The amplitudes of the incoming (towards $x = 0$) and outgoing (towards $|x| = \infty$) waves, $\mathbf{A} = \{A_1, A_2\}$ and $\mathbf{B} = \{B_1, B_2\}$, respectively, are related by

$$\mathbf{B} = (\boldsymbol{\kappa} + i\mathbf{S})^{-1}(\boldsymbol{\kappa} - i\mathbf{S})\,\mathbf{A} \equiv \mathbf{T}\mathbf{A}, \tag{2.8}$$

where
$$\boldsymbol{\kappa} = \begin{bmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix} \tag{2.9a,b}$$

is the scattering matrix, and $\mathbf{T}$ is the transmission matrix; see I, §3, for details. It follows from (3.1) below that $S_{21} = S_{12}$.

The preceding formulation may be extended to oblique incidence as in I.

3. Variational approximations

The elements of the scattering matrix admit the representation [cf. I(5.2), I(2.15), I(2.5), I(2.6)]

$$\frac{1}{S_{mn}}=\frac{\int_0^a\int_0^a u_m(y)\,G(y,\eta)\,u_n(\eta)\,d\eta\,dy}{\int_0^a \chi_m(y)\,u_n(y)\,dy\int_0^a \chi_n(\eta)\,u_m(\eta)\,d\eta}, \tag{3.1}$$

where
$$G(y,\eta)=\sum_{m=1,2}\sum_k k^{-1}\psi_m(y,k)\,\psi_m(\eta,k), \tag{3.2}$$

$$\psi(y,k)=2^{\frac{1}{2}}(h-K^{-1}\sin^2 kh)^{-\frac{1}{2}}\cos[k(h-y)], \tag{3.3}$$

$$k\tan kh+K=0 \quad (0<k_1<k_2<\ldots), \tag{3.4}$$

and u_1 and u_2 satisfy the integral equations I(3.9) with h_1 replaced by a therein; (3.1) is invariant with respect to first-order variations of $u_1(y)$ and $u_2(y)$ about the true solutions of this pair of integral equations.

The aperture functions u_1 and u_2 are linearly independent if $h_1 \neq h_2$; however, the approximation
$$u_m(y)=C_m f(y), \tag{3.5}$$

where $f(y)$ is an appropriate trial function, may be expected to yield rather accurate variational approximations to the S_{mn} (cf. I, §§5, 6). Substituting (3.5) into (3.1) and proceeding as in I, §5, we obtain

$$S_{11}=S_{12}/\lambda N=S_{22}/(\lambda N)^2\equiv \kappa_1/X, \tag{3.6}$$

where
$$\lambda N=\int_0^a \chi_2 f\,dy\Big/\int_0^a \chi_1 f\,dy \tag{3.7}$$

and
$$X=\kappa_1\sum_{m=1,2}\sum_k k^{-1}\left\{\int_0^a f(y)\,\psi_m(y,k)\,dy\Big/\int_0^a f(y)\,\chi_1(y)\,dy\right\}^2. \tag{3.8}$$

Note that (3.8) represents an additive separation of the contributions of the non-propagated modes in $x<0$ ($m=1$) and $x>0$ ($m=2$) to X. Substituting (2.9) and (3.6) into (2.8) and choosing
$$\lambda=(\kappa_2/\kappa_1)^{\frac{1}{2}}, \tag{3.9}$$
we obtain [cf. I(5.8)]
$$\mathbf{T}=\left(\frac{1}{1+N^2-iX}\right)\begin{bmatrix}N^2-1-iX & -2\lambda N\\ -2\lambda^{-1}N & 1-N^2-iX\end{bmatrix}. \tag{3.10}$$

The plane-wave approximation to $\mathbf{T}$, which neglects the non-propagated modes (see I, §4), is obtained by choosing
$$f=\chi_1(y) \tag{3.11}$$
and neglecting X in (3.10). The corresponding variational approximation retains X. The limits $Kh_{1,2}\downarrow 0$ and $Kh_1\downarrow 0$, $Kh_2\uparrow\infty$ are of special interest (see third and fourth sentences in §1).

Substituting (2.6), (3.2), (3.3) and (3.11) into (3.7)–(3.9) and letting $Kh_{1,2}\downarrow 0$ (the long-wave limit), we obtain
$$\lambda=N=(h_1/h_2)^{\frac{1}{4}} \tag{3.12a}$$
and
$$X=(Kh_1)^{\frac{1}{2}}\{\mathscr{S}(\alpha_1)+\mathscr{S}(\alpha_2)\} \quad (\alpha_m\equiv a/h_m) \tag{3.12b}$$

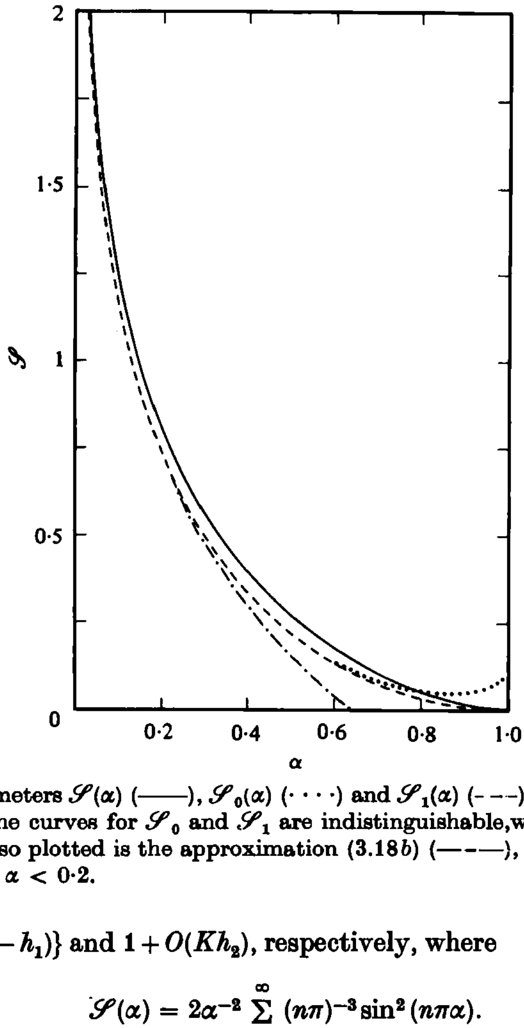

FIGURE 2. The parameters $\mathscr{S}(\alpha)$ (——), $\mathscr{S}_0(\alpha)$ (· · · ·) and $\mathscr{S}_1(\alpha)$ (– – –) as calculated from (3.13), (3.18*a*) and (4.9). The curves for $\mathscr{S}_0$ and $\mathscr{S}_1$ are indistinguishable, within the accuracy of the plot, for $\alpha < 0{\cdot}6$. Also plotted is the approximation (3.18*b*) (—–—), which is indistinguishable from $\mathscr{S}_0$ and $\mathscr{S}_1$ for $\alpha < 0{\cdot}2$.

within $1+O\{K(h_2-h_1)\}$ and $1+O(Kh_2)$, respectively, where

$$\mathscr{S}(\alpha) = 2\alpha^{-2}\sum_{n=1}^{\infty}(n\pi)^{-3}\sin^2(n\pi\alpha). \tag{3.13}$$

The dimensionless parameter $\mathscr{S}(\alpha)$ is plotted in figure 2. It can be shown, by differentiating $\alpha^2\mathscr{S}(\alpha)$ twice, summing the resulting series, and re-integrating, that

$$\mathscr{S}(\alpha) = \frac{2}{\pi}(-\ln 2\pi\alpha+\tfrac{3}{2})-\frac{4}{\pi\alpha^2}\int_0^{\alpha}(\alpha-t)\ln\left(\frac{\sin\pi t}{\pi t}\right)dt \tag{3.14a}$$

$$= (2/\pi)(-\ln 2\pi\alpha+\tfrac{3}{2})+(\pi/18)\alpha^2+O(\alpha^4) \quad (\alpha\downarrow 0) \tag{3.14b}$$

and

$$\mathscr{S}(\alpha) = (2/\pi)(1-\alpha)^2\{-\ln 2\pi(1-\alpha)+\tfrac{3}{2}\}+O\{(1-\alpha)^4\} \quad (\alpha\uparrow 1). \tag{3.14c}$$

The counterparts of (3.12) in the joint limit $Kh_1\downarrow 0$, $Kh_2\uparrow\infty$ (shallow/deep water in $x \lessgtr 0$) are [cf. I, § 6; ψ_2 is given by I(2.9*b*) in the limit $Kh_2\uparrow\infty$, and the summation over k for $m=2$ in (3.8) then must be replaced by integration from $k=0$ to $k=\infty$]

$$\lambda \sim (Kh_1)^{\frac{1}{2}},\quad N\sim 2^{\frac{1}{2}}(Kh_1)^{\frac{1}{4}},\quad X\sim (Kh_1)^{\frac{1}{2}}[\mathscr{S}(\alpha_1)+(2/\pi)\{-\ln 2Ka-\gamma+\tfrac{3}{2}\}], \tag{3.15a, b, c}$$

where $\gamma = 0{\cdot}5772\ldots$ is Euler's constant.

The approximation (3.11) fails to model the singularity at the edge of the barrier. An approximation that does model this singularity in the limit $a \downarrow 0$ is obtained by considering potential flow through the aperture $|y| < a$ in the plane $x = 0$, which leads to

$$f = (a^2 - y^2)^{-\frac{1}{2}} \equiv f_0(y). \tag{3.16}$$

The approximation (3.12a) remains unchanged in the limit $Kh_{1,2} \downarrow 0$, whilst (3.12b) is replaced by

$$X = (Kh_1)^{\frac{1}{2}} \{\mathscr{S}_0(\alpha_1) + \mathscr{S}_0(\alpha_2)\} \quad (Kh_2 \downarrow 0), \tag{3.17}$$

where

$$\mathscr{S}_0(\alpha) = (2/\pi) \sum_{n=1}^{\infty} n^{-1} J_0^2(n\pi\alpha) \tag{3.18a}$$

$$= -(2/\pi) \ln(\pi\alpha/2) + O(\alpha^2) \quad (\alpha \downarrow 0). \tag{3.18b}$$

The approximations (3.15a, b) also remain unchanged in the joint limit $Kh_1 \downarrow 0$, $Kh_2 \uparrow \infty$, whilst (3.15c) is replaced by

$$X \sim (Kh_1)^{\frac{1}{2}} [\mathscr{S}_0(\alpha_1) - (2/\pi)\{\ln(Ka/2) + \gamma\}] \quad (Kh_2 = \infty). \tag{3.19}$$

The dimensionless parameter $\mathscr{S}_0(\alpha)$ is plotted in figure 2. It is similar to, but smaller than, $\mathscr{S}(\alpha)$ for $\alpha < 0{\cdot}8$ ($\mathscr{S} - \mathscr{S}_0 \to 0{\cdot}072$ for $\alpha \downarrow 0$); it differs qualitatively therefrom for $\alpha > 0{\cdot}8$. The variational principle (see §4) implies that (3.17) is more accurate than (3.12b) for $\alpha_1 = \alpha_2 \downarrow 0$ and suggests that (3.17)/(3.19) is more accurate than (3.12b)/(3.15c) for $\alpha_2 < \alpha_1 \downarrow 0$, whilst the converse is true for $\alpha_1 \uparrow 1$; physical considerations imply $X = 0$ for $\alpha_1 = \alpha_2 = 1$, which condition is satisfied by (3.12b) but not by (3.17).

An *ad hoc* approximation that is within 1 % of (3.17) for $0 < \alpha_1 < 0{\cdot}5$, yields $X = 0$ for $\alpha_1 = \alpha_2 = 1$, and appears to be superior (*is* superior if $\alpha_1 = \alpha_2$) to (3.12b) for $0 < \alpha_1 < 1$ is obtained by replacing $\mathscr{S}_0$ in (3.17) by $\mathscr{S}_1$, as given by (4.9) below. Similarly, $\mathscr{S}_0$ may be replaced by $\mathscr{S}_1$ in (3.19).

4. Equal depths

Letting $h_1 = h_2 \equiv h$ in §§2 and 3, we obtain $N = \lambda = 1$,

$$S_{11} = S_{12} = S_{21} = S_{22} \equiv \kappa/X \tag{4.1}$$

and

$$X = 2\kappa \sum_k k^{-1} \left\{ \int_0^a u(y) \psi(y,k)\, dy \middle/ \int_0^a u(y) \chi(y)\, dy \right\}^2, \tag{4.2}$$

where the summation is over the positive roots of (3.4). The variational form (4.2) is an absolute minimum with respect to first-order variations of u about the true solution to the integral equation I(3.9) (after dropping subscripts and replacing h by a therein). Setting $\kappa_1 = \kappa_2 = \kappa$, $A_2 = 0$, $B_1 = RA_1$, and $B_2 = -TA_1$ in (2.8) and invoking (3.10) and (4.1), we obtain

$$T = 1 - R = (1 - \tfrac{1}{2} iX)^{-1} \tag{4.3}$$

for the transmission and reflexion coefficients of the barrier.

Letting $u = C\chi$ and substituting χ and ψ from (2.6) and (3.3), we obtain the variational approximation

$$X \doteqdot \frac{8\beta\{1 + (2\beta)^{-1} \sinh 2\beta\}}{[\alpha + (2\beta)^{-1}\{\sinh 2\beta - \sinh 2\beta(1-\alpha)\}]^2} \times \sum_{\ell} \frac{\{\ell \cosh\beta(1-\alpha) \sin \ell(1-\alpha) + \beta \sinh\beta(1-\alpha) \cos \ell(1-\alpha)\}^2}{\ell(\beta^2 + \ell^2)^2\{1 + (2\ell)^{-1} \sin 2\ell\}}, \tag{4.4}$$

where $$\alpha = a/h, \quad \beta = \kappa h, \quad \ell = kh. \tag{4.5a, b, c}$$

Letting $\beta \downarrow 0$ in (4.4) and (2.7), we obtain

$$X \to 2(Kh)^{\frac{1}{2}} \mathscr{S}(\alpha) \quad (Kh \downarrow 0), \tag{4.6}$$

where $\mathscr{S}(\alpha)$ is given by (3.13).

The approximation (4.6) may be improved by replacing $\mathscr{S}(\alpha)$ by $\mathscr{S}_0(\alpha)$ for $0 < \alpha < 0{\cdot}8$. This corresponds to choosing the trial function $u = Cf_0$ in (4.2). Then, since the variational approximation exceeds the exact result for any u other than the exact solution and since $\mathscr{S}_0 < \mathscr{S}$ in $0 < \alpha < 0{\cdot}8$, the trial function Cf_0 is superior to $C\chi$ in that range. A more accurate result may be obtained by introducing the change of variable

$$\cos(\pi y/h) = \cos^2(\pi\alpha/2) + \sin^2(\pi\alpha/2)\cos\theta \tag{4.7}$$

and the trial function

$$u(y) = C \operatorname{cosec}\theta \sin(\pi y/h), \tag{4.8}$$

which is suggested by a similar treatment of the problem of acoustic diffraction by an aperture in a rectangular wave guide (Miles 1946). Substituting (4.8) into (4.2) and letting $Kh \downarrow 0$, we obtain [see Miles (1946) for analytical details] X in the form (4.6) with $\mathscr{S}$ replaced by

$$\mathscr{S}_1(\alpha) = (2/\pi) \ln \operatorname{cosec}(\pi\alpha/2). \tag{4.9}$$

$\mathscr{S}_1$, which is plotted in figure 2, is smaller than $\mathscr{S}_0$ (so that the corresponding approximation to X is superior to that based on $\mathscr{S}_0$) for $0 < \alpha \leqslant 1$, but the difference is less than 1 % for $\alpha < 0{\cdot}5$, and $\mathscr{S}_0 - \mathscr{S}_1 \downarrow 0$ as $\alpha \downarrow 0$.

Letting $\beta \uparrow \infty$ and replacing the summation by integration in (4.4), we obtain

$$X \sim X_\infty(Ka) \quad (Kh \uparrow \infty), \tag{4.10}$$

where
$$X_\infty(x) = \frac{2x^2}{\pi \sinh^2 x} \int_0^\infty \frac{\sin^2 t \, dt}{t(t^2 + x^2)} \tag{4.11a}$$

$$= (\pi \sinh^2 x)^{-1} \{\gamma + \ln 2x - \tfrac{1}{2} e^{-2x} Ei(2x) + \tfrac{1}{2} e^{2x} E_1(2x)\} \tag{4.11b}$$

$$= (2/\pi)(\tfrac{3}{2} - \gamma - \ln 2x) + O(x^2 \ln x) \quad (x \downarrow 0), \tag{4.11c}$$

and Ei and E_1 are exponential integrals (Abramowitz & Stegun 1964); see figure 3.

Johnson *et al.* (1951), starting from the hypothesis that the incident wave energy above/below the top of the barrier is totally transmitted/reflected, obtain

$$|T| = \left[1 - \left\{\frac{2\beta(1-\alpha) + \sinh 2\beta(1-\alpha)}{2\beta + \sinh 2\beta}\right\}\right]^{\frac{1}{2}}. \tag{4.12}$$

Their limiting results,

$$|T| \to \alpha^{\frac{1}{2}} \quad (Kh \downarrow 0) \quad \text{and} \quad |T| \to (1 - e^{-2Ka})^{\frac{1}{2}} \quad (Kh \uparrow \infty), \tag{4.13a, b}$$

differ substantially from those obtained through the substitution of (4.6) and (4.10) into (4.3). It therefore appears that their hypothesis is untenable.

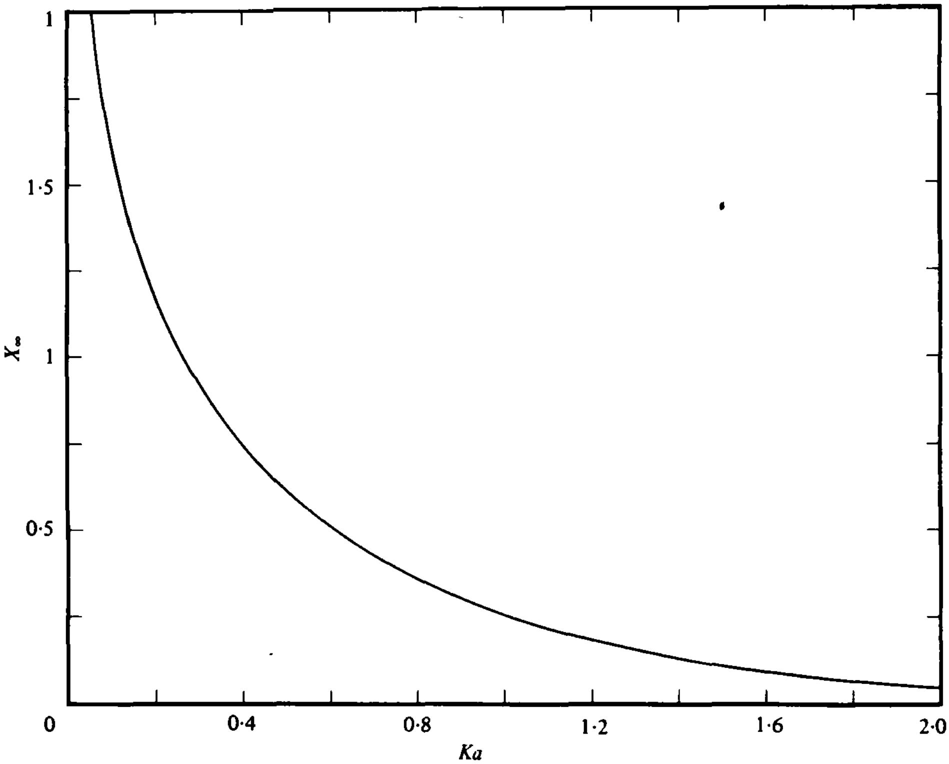

FIGURE 3. The parameter $X_\infty(Ka)$, as given by (4.10) and (4.11).

5. Shelf resonance

The boundary condition (2.4*b*) may be satisfied for $d/h_1 \gg 1$ by choosing (cf. I, §7)

$$A_1 = \tfrac{1}{2}C_1 e^{i\delta}, \quad B_1 = \tfrac{1}{2}C_1 e^{-i\delta}, \quad \delta = \kappa_1 d. \tag{5.1a, b, c}$$

Substituting (5.1) into (2.8) and solving for B_2/A_2 and C_1/A_2, we obtain the reflexion coefficient

$$R \equiv \frac{B_2}{A_2} = e^{2i\tau}, \quad \tau = \tan^{-1}\left\{\frac{\kappa_1 S_{22}\tan\delta - |\mathbf{S}|}{\kappa_2(S_{11} - \kappa_1 \tan\delta)}\right\} \tag{5.2}$$

and the free-surface-displacement transmission coefficient

$$T \equiv -C_1 \frac{\chi_1(0)}{\chi_2(0)} = \frac{2\kappa_2 S_{12}\chi_{12}}{\kappa_2(S_{11}\cos\delta - \kappa_1 \sin\delta) - i(\kappa_1 S_{22}\sin\delta - |\mathbf{S}|\cos\delta)}, \tag{5.3}$$

where

$$\chi_{12} = \chi_1(0)/\chi_2(0), \quad |\mathbf{S}| = S_{11}S_{22} - S_{12}^2. \tag{5.4a, b}$$

Resonance may be defined as that condition for which $\tau = \frac{1}{2}\pi$ (or, more generally, $\tau = \frac{1}{2}\pi + n\pi$, $n = 0, 1, 2, \ldots$), which implies

$$\delta + \tan^{-1}X = \tfrac{1}{2}\pi \quad (X \equiv \kappa_1/S_{11}). \tag{5.5}$$

It follows from (5.5) that the effect of the non-propagated modes is equivalent to an incremental shelf length of

$$d_1 = \kappa_1^{-1}\tan^{-1}X \to 1/S_{11} \quad (Kh_1 \downarrow 0). \tag{5.6}$$

The corresponding Q (ratio of the resonant frequency to the half-power bandwidth of the resonance curve) is

$$Q = \tfrac{1}{2}\kappa_2\{d(S_{11}^2+\kappa_1^2)+S_{11}\}/S_{12}^2. \quad (5.7)\dagger$$

The approximation (3.5), followed by the substitution of (3.6) and (3.9) into (5.3) and (5.7), yields

$$T = 2\lambda N(\cos\delta - X\sin\delta - iN^2)^{-1}\chi_{12} \quad (5.8)$$

and

$$Q = \tfrac{1}{2}N^{-2}\{X+\delta(1+X^2)\}^{-1}. \quad (5.9)$$

The limiting results for $Kh_{1,2}\downarrow 0$ may be obtained by substituting (3.12a) and either (3.12b) or (3.17) into (5.6), (5.8) and (5.9) and similarly for $Kh_1\downarrow 0$ and $Kh_2\uparrow\infty$, using (3.15a, b) and either (3.15c) or (3.19). If $h_1 = h_2$, $\lambda = N = 1$, and X should be calculated from (4.6) and (4.9).

6. Comparison with Tuck (1980)

After completing the preceding work, I learned of Tuck's (1980) work on the problem of shelf resonance. Tuck assumes $Kh_1 \ll 1$ and either $Kh_2 \ll 1$ or $Kh_2 = \infty$ and matches outer solutions, of the form (2.5) above, in $x < 0$ and $x > 0$ to an inner solution in the neighbourhood of the barrier, where he replaces the free surface by a rigid boundary, invokes potential theory, and solves the reduced boundary-value problem by conformal mapping. His results imply

$$X = 2C(Kh_1)^{\frac{1}{2}} \quad (Kh_2 \ll 1), \quad (6.1)$$

where C is his 'blockage coefficient' and is given by his (A 14), and

$$X = (2/\pi)(Kh_1)^{\frac{1}{2}}(C_\infty - \ln kh_1 - \gamma) \quad (Kh_1 \ll 1,\ Kh_2 = \infty), \quad (6.2)$$

where C_∞ is given by his (A 24). The approximation (6.1) is to be compared with either (3.12b) or (3.17), whilst (6.2) is to be compared with either (3.15c) or (3.19); (6.1)/(6.2) is equivalent to (3.17)/(3.19) in the (narrow gap) limit $\alpha_1\downarrow 0$.

If $h_1 = h_2 = h$, Tuck's blockage coefficient may be reduced to [after setting $\mu = 1$ and $\beta = \sec\frac{1}{2}\pi\alpha$ in his (A 14)] $C = \mathscr{S}_1(\alpha)$, and (6.1) then is identical with the approximation implied by (4.6) and (4.9).

It seems likely that Tuck's solution for $h_1 \neq h_2$ models the singularity at the edge of the barrier better than does either (3.11) or (3.16), and hence that (6.1) is slightly superior to either (3.12b) or (3.15c). On the other hand, Tuck's solution is valid only for $Kh_1 \ll 1$ and either $Kh_2 \ll 1$ or $Kh_2 \gg 1$ (in particular, his replacement of the free surface by a rigid boundary appears to be appropriate only for $Kh_1 \ll 1$), whereas the variational approximations in §§3 and 4 above are valid for arbitrary $Kh_{1,2}$ (cf. I).

This work was carried out during my tenure as an Overseas Fellow of Churchill College and a visitor in the Department of Applied Mathematics and Theoretical Physics of the University of Cambridge. It was partially supported by the Physical Oceanography Division, National Science Foundation (NSF Grant OCE77–24005) and by a contract with the Office of Naval Research.

† This is actually the Q for a resonance curve of amplitude *vs.* κ_1 on the assumption that the variation of S_{11} with κ_1 over the resonant peak is negligible. It corresponds to the usual Q if $Kh_1 \ll 1$.

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Note added in proof (29 April 1981).

The limiting result $|R| \rightarrow \kappa h \mathscr{S}_1(\kappa h)$ as $\kappa h \rightarrow 0$ with $h_1 = h_2 = h$, as implied by (4.3), (4.6) and (4.9) was obtained by Kreisel (1949).

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